Numerical Integration based on Interpolation

Idea: some functions are easy to integrate analytically, e.g. $f(x) = e^x$
$f(x) = \sin x$

others, like $f(x) = e^{-x^2}$ or $f(\theta) = \int_0^\pi \sin(\sin\theta)\,d\theta$ may not be.

Goal: Want a method to approximate

$\int_a^b f(x)\,dx$ by using only $f(x_i)$, $i = 0, \dots, n$.

Reasonable approach: use polynomial interp. to write $f(x) \approx p(x) = \underbrace{\sum_{i=0}^{n} f(x_i)\,\ell_i(x)}_{\text{Lagrange form of interp. poly.}}$

Then "hope" that $\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$

So $\int_a^b f(x)\,dx \approx \int_a^b \sum_{i=0}^n f(x_i)\,\ell_i(x)\,dx$

$$= \sum_{i=0}^n \int_a^b f(x_i)\,\ell_i(x)\,dx$$

exchange order of Σ & $\int$

$$= \sum_{i=0}^n f(x_i) \underbrace{\int_a^b \ell_i(x)\,dx}_{\text{call this } A_i}$$

$f(x_i)$ is a number

$$= \sum_{i=0}^n A_i f(x_i)$$

called Newton-Coates formula when the x_i are equispaced

In fact, you already know some examples of this

Trapezoid rule: when $n=1$ & $x_0=a, x_1=b$

we have $\ell_0(x) = \dfrac{x_1 - x}{x_1 - x_0} = \dfrac{b-x}{b-a}$

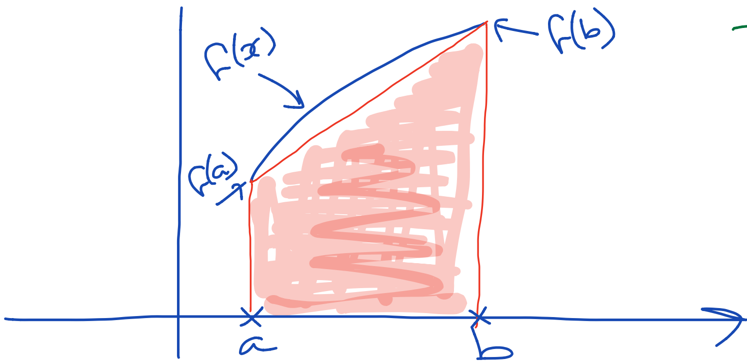
$$\ell_1(x) = \frac{x_0 - x}{x_0 - x_1} = \frac{x-a}{b-a}$$

$$\Rightarrow A_0 = \int_a^b \frac{b-x}{b-a}\,dx = \frac{b(b-a)}{b-a} - \frac{b^2-a^2}{2(b-a)} = b - \frac{b+a}{2}$$

$$A_0 = \frac{b-a}{2}$$

$$A_1 = \frac{b-a}{2} \text{ (similarly)}$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$$



approximate the area under $f(x)$ by the area of the trapezoid.

Its error term is $-\frac{1}{12}(b-a)^3 f''(\xi)$ for some $\xi \in (a,b)$

(Proof uses MVT for integrals)

Composite trapezoid rule: (1) Subdivide $[a,b]$ into n pieces using $a = x_0 < x_1 < x_2 < \cdots < x_n < b$

(2) use the trapezoid rule for each piece

$$\Rightarrow \int_a^b f(x) = \sum_{i=1}^{n} \underbrace{\int_{x_{i-1}}^{x_i} f(x)\,dx}_{\approx \frac{(x_i - x_{i-1})}{2}\left[f(x_i) + f(x_{i-1})\right]}$$

$\Rightarrow$

$$\int_a^b f(x)\,dx \approx \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1})\left(f(x_{i-1}) + f(x_i)\right)$$

Comp. Trap. rule with equal spacing

when all the subintervals of $[a,b]$ are the same length $h = x_i - x_{i-1}$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right]$$

Err term $= -\frac{1}{12}(b-a)h^2 f''(\xi)$, $\xi \in (a,b)$

—X—

Back to the general, non-composite, case

Recall:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

$$A_i \to \int_a^b \ell_i(x)\,dx$$

Since for polynomials of degree $\leq n$

$$f(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x)$$ then our

integration formula is exact for poly. of deg. $\leq n$.

This observation allows us to find the A_i's "easily" by the method of undetermined coefficients

Example: $n=2$, $[a,b]=[0,1]$ & $x_0=0, x_1=\frac{1}{2}$
$x_2=1$

$$\Rightarrow \int_a^b f(x) \approx A_0 f(0) + A_1 f(1) + A_2 f(2)$$

Formula is exact for poly. of degree ≤ 2

$$\text{So } \underbrace{\int_0^1 1\,dx}_{1} = A_0 + A_1 + A_2$$

$f(x)$

$$\underbrace{\int_0^1 x\,dx}_{1/2} = A_0 \cdot 0 + A_1 \cdot \frac{1}{2} + A_2$$

$f(x)$

$$\underbrace{\int_0^1 x^2\,dx}_{1/3} = A_0 \cdot 0 + A_1/4 + A_2$$

3 eq'n & 3 unknowns $\Rightarrow$ $A_1 = \frac{2}{3}$, $A_2 = \frac{1}{6} = A_0$

& $\int_0^1 f(x)\,dx \approx \frac{1}{6} f(0) + \frac{2}{3} f(\frac{1}{2}) + \frac{1}{6} f(1)$

Simpson's Rule

Repeat the same calculation but with arbitrary $[a,b]$ & $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

to get

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)}$$

Simpson's rule

Remark: should be exact for poly. of degree ≤ 2

but is also exact for poly. of degree ≤ 3

err term $= -\frac{1}{90}\left[\frac{b-a}{2}\right]^5 f^{(4)}(\xi)$, $\xi \in (a,b)$

Composite simpson's rule can also be used (see book)

Error Analysis

Want an expression for the error in numerical integrat-ion; that is, we want an expression for

$$\int_a^b f(x) - \sum_{i=0}^{N} A_i f(x_i)$$

Recall: $A_i = \int_a^b \ell_i(x)\,dx$ where $\ell_i(x)$ comes from the Lagrange interp. poly:

$$P(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x)$$

Also: $$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

So: $$\int_a^b f(x) - \sum_{i=0}^{n} A_i f(x_i) = \frac{1}{(n+1)!} \int_a^b \underbrace{f^{(n+1)}(\xi_x)} \prod_{i=0}^{n} (x - x_i)\,dx$$

can't do much about this as it depends on the f$^{\underline{t}}$ we're numerically integrating. Reasonable to just use an upper bound

$$M = \max_{x \in [a,b]} |f^{(n+1)}(x)|$$

or $M \geq \ldots$

Then $$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{n} A_i f(x_i) \right| \leq \frac{M}{(n+1)!} \int_a^b \prod_{i=0}^{n} |x - x_i|\,dx$$